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Exactly solvable non-linear generalisations of the Jaynes–Cummings model

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Abstract. Exactly solvable generalisations of the Jaynes-Cummings model are considered which are non-linear both in bosonic and spin variables. Exact wavefunctions and energy levels are found for the corresponding systems.

1. Introduction

Studies of the quantum model of interaction of a $(2r+1)$-level atom, spin $s=r$, with a radiation resonance field were initiated by Jaynes and Cummings (1963), who first succeeded in obtaining an exact solution to a Dicke model with the value of $r=\frac{1}{2}$. Subsequently, various versions of multiboson processes were studied in two-level atoms. In particular, Buck and Sukumar (1981a, b, 1984a) have found an exact solution to the equations of motion for an atomic system with the interaction non-linear in bosonic variables. From a physical point of view, a distinctive feature of the Jaynes-Cummings model and its multiboson modifications is the periodic reproduction of atomic energy oscillations due to the initial coherent pumping, observed for the first time by Eberly *et al* (1981).

In this paper, exactly solvable generalisations of the Jaynes-Cummings model which are non-linear in bosonic and spin variables are considered. Owing to the conservation laws, the state space of a system can be decomposed into a direct sum of finite-dimensional subspaces corresponding to fixed values of constants of motion. An eigenstate vector of a system parametrised by a fixed set of these constants is determined as an expansion over the basis of an appropriate finite-dimensional subspace and the expansion coefficients are defined from the Schrödinger equation that is reduced on every subspace to a finite system of algebraic equations.

In § 2 a multiboson variant of the one-mode Jaynes-Cummings model is formulated; the structure of state space is thoroughly analysed, and exact wavefunctions and energy levels of a system are found. In § 3 the same is done for a two-mode version of the Jaynes-Cummings non-linear problem and the completeness of the system of eigenfunctions obtained is discussed. Section 4 is devoted to the consideration of interactions that are non-linear both in bosonic and spin variables.

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2. The one-mode non-linear Jaynes–Cummings problem

Consider the Hamiltonian

$$\mathcal{H} = \omega a^+ a + \omega_0 S_3 + \lambda (a^+)^k a^l f(a^+ a) S_+ + \lambda f(a^+ a)(a^+)^l a^k S_- \tag{1}$$

describing the interaction of an atomic system with $(2r+1)$ equidistant levels with a one-mode radiation field a: $[a, a^+] = 1$. Generators S of a $(2r+1)$-dimensional representation of the SU(2) group obey the relations $[S_3, S_\pm] = \pm S_\pm$, $[S_+, S_-] = 2S_3$ where the operators $S_\pm = S_1 \pm iS_2$ increase (decrease) the energy of an atom by ω_0. As is seen from (1), transitions between neighbouring levels proceed through the emission of l or k phonons and absorption of k or l phonons, and the intensity of interaction $\lambda f(a^+ a)$ depends on the intensity of a phonon field by the function f satisfying the condition $f(a^+ a)|n\rangle = f(n)|n\rangle$ where $a^+ a|n\rangle = n|n\rangle$. Model (1) at $k = 0$, $l = 1$, $f \equiv 1$, $r = \frac{1}{2}$ reduces to the Jaynes-Cummings problem whose exact solution has been known since 1963 (Jaynes and Cummings 1963, Rupasov 1982, Lee 1973).

2.1. The structure of state space

The state space of system (1) is generated by a basis $\{|\phi_m^n\rangle = |n\rangle|m\rangle;\ n \geq 0, -r \leq m \leq r\}$ where $S_3|m\rangle = m|m\rangle$. Conservation of the charge of a system $\hat{N} = a^+ a + (l-k)S_3$, $[\mathcal{H}, \hat{N}] = 0$ limits the range of variation of n and m by the condition $N = n + (l-k)m$, where N is an eigenvalue of the operator $\hat{N}$. In this way, the state space can be represented by $\Sigma \oplus \mathcal{H}_N$, where every subspace $\mathcal{H}_N$ corresponding to a particular value of the charge N is generated by the basis

$$\{|\phi_m^N\rangle = |N-(l-k)m\rangle|m\rangle\} \qquad \hat{N}|\phi_m^N\rangle = N|\phi_m^N\rangle \tag{2}$$

where m (at $l > k$) and $-m$ (at $l < k$) vary from $-r$ to $\min(r, m_{\max})$ and $m_{\max}$ is defined from the conditions

$$\frac{N}{|l-k|} - 1 < m_{\max} \leq \frac{N}{|l-k|} \qquad m_{\max} \in \{-r+p,\ p = 0, 1, 2, \ldots\}$$

with the dimensionality of a subspace $\mathcal{H}_N$

$$\dim \mathcal{H}_N = 1 + r + \min(r, m_{\max}).$$

The charge eigenvalues take the values

$$N = n - |l-k|r;\ n \geq 0. \tag{3}$$

As a result, $m_{\max} = [n/|l-k|] - r$, where $[x]$ is the integer part of x and

$$\dim \mathcal{H}_{N=n-|l-k|r} = 1 + r + \min\left(r, \left[\frac{n}{|l-k|}\right] - r\right) \qquad l \neq k. \tag{4}$$

At $l = k$ the dimensionality of any subspace $\mathcal{H}_N$ is $2r+1$. Let $l > k$ for definiteness; then the vector $|\psi^N\rangle$ obeying the Schrödinger equation

$$\mathcal{H}|\psi\rangle = \mathcal{E}|\psi\rangle \tag{5}$$

with Hamiltonian (1) and corresponding to the charge eigenvalue N

$$\hat{N}|\psi^N\rangle = N|\psi^N\rangle$$

can be represented by the following expansion over the basis of subspace $\mathcal{H}_N$ (2):

$$|\psi^{N=n-|l-k|r}\rangle = \sum_{m=-r}^{\bar{m}} c_m |\phi_m^N\rangle = \sum_{j=0}^{r+\bar{m}} \tilde{c}_j |\phi_{-r+j}^N\rangle \tag{6}$$

where

$$\bar{m} = \min(r, m_{\max}) = \min\left(r, \left[\frac{n}{|l-k|}\right] - r\right). \tag{7}$$

2.2. *Energy levels and eigenfunctions at* $r = \frac{1}{2}$

Consider the Hamiltonian (1) at $r = \frac{1}{2}$ and $l > k$. Eigenvalues of the operator $\hat{N}$ take the values $N = n - (l-k)/2$; $n \geq 0$, according to (3). Expansion (6) in this case is as follows:

$$|\psi^{N=n-(l-k)/2}\rangle \equiv |\psi_n\rangle = \sum_{j=0}^{1/2+\min(1/2,[n/(l-k)]-1/2)} \tilde{c}_j |\phi_{-1/2+j}^N\rangle \tag{8}$$

or

$$|\psi_n\rangle = (1+\alpha^2)^{-1/2}(|n\rangle|-\tfrac{1}{2}\rangle + \alpha|n-l+k\rangle|\tfrac{1}{2}\rangle) \qquad n \geq l-k$$

$$|\psi_n\rangle = |n\rangle|-\tfrac{1}{2}\rangle \qquad 0 \leq n < l-k$$

where α is a parameter expressed through $\tilde{c}_j$ and defined by equation (5). Inserting (8) into the Schrödinger equation (5) we obtain the following solutions.

For $n \geq l$:

$$|\psi_n^{(\pm)}\rangle = (1+\alpha_{nkl}^{(\pm)2})^{-1/2}\begin{pmatrix} \alpha_{nkl}^{(\pm)}|n-l+k\rangle \\ |n\rangle \end{pmatrix} \tag{9a}$$

$$\mathscr{E}_n^{(\pm)} = \omega n - \tfrac{1}{2}\omega(l-k) \pm \sqrt{\Omega_{kl}^2 + \lambda^2 c_{nkl}^2 f^2(n)} \tag{9b}$$

where

$$\alpha_{nkl}^{(\pm)} = \frac{-\Omega_{kl} \pm (\Omega_{kl}^2 + \lambda^2 c_{nkl}^2 f^2(n))^{1/2}}{\lambda f(n) c_{nkl}} \qquad \Omega_{kl} = \tfrac{1}{2}[\omega(l-k) - \omega_0]$$

$$c_{nkl}^2 = \frac{n!(n-l+k)!}{(n-l)!(n-l)!}$$

and

$$|n\rangle|-\tfrac{1}{2}\rangle = \begin{pmatrix} 0 \\ |n\rangle \end{pmatrix} \qquad |n\rangle|\tfrac{1}{2}\rangle = \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix}.$$

For $l-k \leq n < l$:

$$|\psi_n^{(+)}\rangle = \begin{pmatrix} 0 \\ |n\rangle \end{pmatrix} \qquad |\psi_n^{(-)}\rangle = \begin{pmatrix} |n-l+k\rangle \\ 0 \end{pmatrix} \tag{10a}$$

$$\mathscr{E}_n^{(+)} = \omega n - \omega_0/2 \qquad \mathscr{E}_n^{(-)} = \omega(n-l+k) + \omega_0/2. \tag{10b}$$

Finally, for $0 \leq n < l-k$ we obtain

$$|\psi_n\rangle = \begin{pmatrix} 0 \\ |n\rangle \end{pmatrix} \qquad \mathscr{E}_n = \omega n - \omega_0/2. \tag{11a, b}$$

From (9*a*)-(11*a*) it follows that $\dim \mathscr{H}_{N;n\geqslant l-k}=2$, $\dim \mathscr{H}_{N,n<l-k}=1$ in full correspondence with the general formula (4) if $r=\frac{1}{2}$. The case $k>l$, $k=l$ is treated in an analogous manner.

Formulae (9)-(11) at $f\equiv 1$, $k=0$, $l=1$ become the results of the standard Jaynes-Cummings model. Expressions obtained by Buck and Sukumar (1981a) for various averages of the energy operator of an atomic system $S_3(t)=e^{i\mathscr{H}t}S_3(0)\,e^{-i\mathscr{H}t}$ follow from the above formulae if one sets $f(x)=\sqrt{x}$, $k=0$, $l=1$. Note also that for $f=1$, $k=0$ and an arbitrary l, Buck and Sukumar (1981b) have found squares of normal modes of oscillations of an atomic system $\omega_n^2=(\mathscr{E}_n^{(+)}-\mathscr{E}_n^{(-)})^2$ which are shown to represent eigenvalues of the squared frequency operator $\hat{\Omega}^2$ in the equation $\ddot{S}_3+\hat{\Omega}^2S_3=A(t)$. Using formulae (9*b*)-(11*b*) we arrive at the results obtained by Buck and Sukumar (1981b).

3. The two-mode non-linear Jaynes–Cummings problem

Now consider a generalisation of the Jaynes-Cummings model to the case of the interaction of a $(2r+1)$-level system with a two-mode field a, b:

$$\mathscr{H}=\omega_1a^+a+\omega_2b^+b+\omega_0S_3+\lambda a^l(b^+)^kS_++\lambda(a^+)^lb^kS_-. \tag{12}$$

This model possesses two independent constants of motion:

$$\hat{N}=a^+a+lS_3 \qquad \hat{M}=b^+b-kS_3.$$

The space state of system (12) is generated by a set $\{|\phi_m^{n,p}\rangle=|n\rangle|p\rangle|m\rangle;\ n,p\geqslant 0,\ -r\leqslant m\leqslant r\}$ where $a^+a|n\rangle=n|n\rangle$, $b^+b|p\rangle=p|p\rangle$, $S_3|m\rangle=m|m\rangle$. Owing to the conservation laws, $N=n+lm$ and $M=p-km$ where N and M are common eigenvalues of charges $\hat{N}$ and $\hat{M}$. Every subspace $\mathscr{H}_{N,M}$ corresponding to fixed values of the charges is generated by the basis

$$\{|\phi_m^{N,M}\rangle=|N-lm\rangle|M+km\rangle|m\rangle\} \tag{13}$$
$$\hat{N}|\phi_m^{N,M}\rangle=N|\phi_m^{N,M}\rangle \qquad \hat{M}|\phi_m^{N,M}\rangle=M|\phi_m^{N,M}\rangle.$$

It may be verified that feasible common eigenvalues of charges form the sets

$$\{N=n-lr,\ M=p+kr;\ n,p\geqslant 0\} \qquad \{N=n+lr,\ M=p-kr;\ n,p\geqslant 0\}.$$

The dimensionality of subspace can be established (cf § 2):

$$\begin{aligned}&\dim \mathscr{H}_{N,M}=1+r+\min\{r,[n/l]-r\} && N=n-lr && M=p+kr\\ &\dim \mathscr{H}_{N,M}=1+r+\min\{r,[p/k]-r\} && N=n+lr && M=p-kr<kr\end{aligned} \tag{14}$$

and the expansion can be written for the solution to the Schrödinger equation (5) corresponding to the eigenvalues of charges N and M over the basis (13):

$$\begin{aligned}&|\psi^{N,M}\rangle=\sum_{j=0}^{r+\tilde{m}_1}c_j|\phi_{-r+j}^{N,M}\rangle && N=n-lr && M=p+kr\\ &|\psi^{N,M}\rangle=\sum_{j=0}^{r+\tilde{m}_2}\tilde{c}_j|\phi_{r-j}^{N,M}\rangle && N=n+lr && M=p-kr<kr\end{aligned} \tag{15}$$

where

$$\tilde{m}_1=\min\{r,[n/l]-r\} \qquad \tilde{m}_2=\min\{r,[p/k]-r\} \qquad l,k\neq 0.$$

3.1. Energy levels and eigenfunctions at $r=\frac{l}{2}$

At $r=\frac{1}{2}$ expansion (15) assumes the form

$$|\psi^{N=n-l/2,M=p+k/2}\rangle \equiv |\psi_{np}\rangle = (1+\alpha^2)^{-1/2}(|n\rangle|p\rangle|-\tfrac{1}{2}\rangle + \alpha|n-l\rangle|p+k\rangle|\tfrac{1}{2}\rangle) \qquad (16)$$

$n \geqslant l,\ p \geqslant 0$

$$|\psi_{np}\rangle = |n\rangle|p\rangle|-\tfrac{1}{2}\rangle \qquad n < l,\ p \geqslant 0$$
$$|\psi^{N=n+l/2,M=p-k/2<k/2}\rangle \equiv |\tilde{\psi}_{np}\rangle = |n\rangle|p\rangle|\tfrac{1}{2}\rangle \qquad n \geqslant 0,\ p < k. \qquad (17)$$

Inserting (16) and (17) into the Schrödinger equation (5) we obtain the following results.

(i) When $n \geqslant l$ and $p \geqslant 0$

$$|\psi_{np}^{(\pm)}\rangle = (1+\alpha_{np}^{(\pm)2})^{-1/2}\begin{pmatrix}\alpha_{np}^{(\pm)}|n-l\rangle|p+k\rangle\\ |n\rangle|p\rangle\end{pmatrix} \qquad (18a)$$

$$\mathscr{E}_{np}^{(\pm)} = \omega_1 n - \tfrac{1}{2}\omega_1 l + \omega_2 p + \tfrac{1}{2}\omega_2 k \pm (\Omega_{kl}^2 + \lambda^2 c_{np}^2)^{1/2} \qquad (18b)$$

where

$$\alpha_{np}^{(\pm)} = \frac{\Omega_{kl} \pm (\Omega_{kl}^2 + \lambda^2 c_{np}^2)^{1/2}}{\lambda c_{np}} \qquad \Omega_{kl} = \tfrac{1}{2}(\omega_0 - \omega_1 l + \omega_2 k) \qquad c_{np}^2 = \frac{n!(p+k)!}{(n-l)!p!}.$$

(ii) For $n < l$ and $p \geqslant 0$

$$|\psi_{np}\rangle = \begin{pmatrix}0\\ |n\rangle|p\rangle\end{pmatrix} \qquad (19a)$$

$$\mathscr{E}_{np} = \omega_1 n + \omega_2 p - \omega_0/2. \qquad (19b)$$

(iii) For $p < k$ and $n \geqslant 0$

$$|\tilde{\psi}_{np}\rangle = \begin{pmatrix}|n\rangle|p\rangle\\ 0\end{pmatrix} \qquad (20a)$$

$$\tilde{\mathscr{E}}_{np} = \omega_1 n + \omega_2 p + \omega_0/2. \qquad (20b)$$

From (18*a*)-(20*a*)

$$\dim \mathscr{H}_{N,M} = 2 \qquad N = n - l/2 \qquad M = p + k/2 \qquad n \geqslant l,\ p \geqslant 0$$
$$\dim \mathscr{H}_{N,M} = 1 \qquad N = n - l/2 \qquad M = p + k/2 \qquad n < l,\ p \geqslant 0$$
$$\dim \mathscr{H}_{N,M} = 1 \qquad N = n + l/2 \qquad M = p - k/2 \qquad p < k,\ n \geqslant 0$$

in full accordance with the general formula (14) if one sets $r=\frac{1}{2}$ there.

3.2. Completeness of the system of eigenfunctions

Let us verify the completeness of the system of functions (18*a*)-(20*a*). Composing the operator $\Sigma_{\{i\}}|\psi_i\rangle\langle\psi_i|$ we obtain a 2×2 matrix

$$\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

where, for instance,

$$a_{11} = \sum_{n\geqslant l,p\geqslant 0} \frac{\alpha_{np}^{(+)2}}{1+\alpha_{np}^{(+)2}} |n-l\rangle\langle n-l||p+k\rangle\langle p+k| + \sum_{n\geqslant l,p\geqslant 0} \frac{\alpha_{np}^{(-)2}}{1+\alpha_{np}^{(-)2}} |n-l\rangle\langle n-l||p+k\rangle\langle p+k| + \sum_{n\geqslant 0,p<k} |n\rangle\langle n||p\rangle\langle p|.$$

Owing to the relationship

$$\alpha_{np}^{(+)2}/1+\alpha_{np}^{(+)2} + \alpha_{np}^{(-)2}/1+\alpha_{np}^{(-)2} = 1 \qquad a_{11} = \sum_0^\infty |n\rangle\langle n| \sum_0^\infty |p\rangle\langle p| = 1.$$

Analogously, $a_{22}=1$, $a_{12}=a_{21}=0$. In a similar way, completeness is proved for the system of functions (9*a*)-(11*a*).

4. Generalisations of the Jaynes–Cummings model non-linear in spin and bosonic variables

Consider a generalisation of the Jaynes-Cummings model non-linear in spin and bosonic variables:

$$\mathcal{H}(r;\, s) = \omega a^+ a + \omega_0 S_3 + \sum_{j=1}^{2r} \frac{\lambda_j}{j!}(aS_+)^j + \sum_{j=1}^{2r} \frac{\lambda_j}{j!}(a^+ S_-)^j \tag{21}$$

where r and s are positive integers or half-integers and $S_3|m\rangle = m|m\rangle$, $-s \leqslant m \leqslant s$. The Hamiltonian $\mathcal{H}(r=\frac{1}{2};\, s=\frac{1}{2}) = \omega a^+ a + \omega_0 S_3 + \lambda a S_+ + \lambda a^+ S_-$ is a standard Jaynes-Cummings Hamiltonian and $\mathcal{H}(r=\frac{1}{2};\, s=1)$ represents a Jaynes-Cummings problem for a three-level atom. Exact results for the Hamiltonians $\mathcal{H}(r=\frac{1}{2},\, s=1)$ and $\mathcal{H}(r=\frac{1}{2},\, s=\frac{3}{2})$ are given in Buck and Sukumar (1984b) and Sentitzky (1971). Due to the relations $(S_+)^{2s+1} = (S_-)^{2s+1} = 0$, $\mathcal{H}(r;\, s\leqslant r) = \mathcal{H}(s;\, s)$. On the other hand, $\mathcal{H}(r;\, s>r)$ can be treated as $\mathcal{H}(s;\, s)$, where $\lambda_j = 0$ for $j > 2r$. So it suffices to consider the Hamiltonian $\mathcal{H}(r;\, s=r) \equiv \mathcal{H}_r$.

4.1. Structure of the state space

The Hamiltonian $\mathcal{H}_r$ commutes with the charge operator $\hat{N} = a^+ a + S_3$, the form of which coincides with (2) at $k=0$, $l=1$. Using the results of § 2 we obtain the system of bases of the subspace $(\mathcal{H}_r)_N$ with a distinguished charge:

$$\{|\phi_m^N\rangle = |N-m\rangle|m\rangle\} \qquad \hat{N}|\phi_m^N\rangle = N|\phi_m^N\rangle \tag{22}$$

where N runs over the sequence $N = n-r$; $n \geqslant 0$. Formula (4) reduces to

$$\dim(\mathcal{H}_r)_N = 1 + r + \min(r;\, n-r). \tag{23}$$

The solution to the Schrödinger equation $|\psi^N\rangle$ with Hamiltonian $\mathcal{H}_r$ belonging to the subspace $(\mathcal{H}_r)_N$ can be represented by

$$|\psi^N\rangle = \sum_{j=0}^{r+\min(r,n-r)} \alpha_j |\phi_{-r+j}^N\rangle \tag{24}$$

where the expansion coefficients α_j are defined by (5).

4.2. Wavefunctions and energy levels of the Hamiltonian $\mathcal{H}_1$

According to (21)

$$\mathcal{H}_1 \equiv \mathcal{H}(1;1) = \omega a^+ a + \omega_0 S_3 + \lambda_1 a S_+ + \lambda_1 a^+ S_- + (\lambda_2/2!)(aS_+)^2 + (\lambda_2/2!)(a^+ S_-)^2$$

$$S_3|m\rangle = m|m\rangle \qquad m = 0, \pm 1. \tag{25}$$

The model Hamiltonian (25) describes a three-level atom interacting with a one-mode radiation field; transitions between neighbouring levels are realised by a one-boson exchange whereas between two extreme ones they are by two-boson processes. The intensity of the transitions is different and is determined by the coupling constants λ_1 and λ_2.

Expansion (24) at $r = 1$ is

$$|\psi^{N=n-1}\rangle \equiv |\psi_n\rangle = (1+\alpha_1^2+\alpha_2^2)^{-1/2}(|n\rangle|-1\rangle + \alpha_1|n-1\rangle|0\rangle + \alpha_2|n-2\rangle|1\rangle) \qquad n \geqslant 2 \tag{26}$$

$$|\psi_{n=1}\rangle = (1+\alpha^2)^{-1/2}(|1\rangle|-1\rangle + \alpha|0\rangle|0\rangle) \tag{27}$$

$$|\psi_{n=0}\rangle = |0\rangle|-1\rangle. \tag{28}$$

Inserting (26) into (5), making the change $\alpha_1 \to \sqrt{2n}\,\alpha_1$, $\alpha_2 \to \sqrt{n(n-1)}\,\alpha_2$ and setting $\omega = \omega_0$, we obtain

$$\alpha_1 = \frac{\lambda_1\lambda_2 + \lambda_1 y}{2\lambda_1^2 + ny\lambda_2} \qquad \alpha_2 = \frac{ny - 2\lambda_1^2}{(n-1)(2\lambda_1^2 + ny\lambda_2)} \tag{29}$$

$$\mathcal{E} = \omega(n-1) + y \tag{30}$$

where y obeys the equation

$$ny^3 - y(2\lambda_1^2(2n-1) + n(n-1)\lambda_2^2) - 4\lambda_1^2\lambda_2(n-1) = 0$$

the solution of which is given by

$$y_1 = 2\sqrt{-p/3}\cos\beta/3 \qquad y_{2,3} = -2\sqrt{-p/3}\cos(\beta/3 \pm \pi/3)$$

$$\cos\beta = -q/2\sqrt{-(p/3)^3}$$

with

$$p = -\frac{2\lambda_1^2(2n-1) + n(n-1)\lambda_2^2}{n^2} \qquad q = -\frac{4(n-1)\lambda_1^2\lambda_2}{n^2} \qquad n \geqslant 2. \tag{31}$$

Inserting (27) into (5) we obtain

$$|\psi_{n=1}^{(\pm)}\rangle = (1/\sqrt{2})(|1\rangle|-1\rangle \pm |0\rangle|0\rangle) \qquad \lambda_1 \neq 0 \tag{32a}$$

$$\mathcal{E}_1^{(\pm)} = \pm\sqrt{2}\lambda_1 \qquad \lambda_1 \neq 0 \tag{32b}$$

$$|\psi_1^{(+)}\rangle = |1\rangle|-1\rangle \qquad |\psi_1^{(-)}\rangle = |0\rangle|0\rangle \qquad \lambda_1 = 0 \tag{33a}$$

$$\mathcal{E}_1^{(+)} = \mathcal{E}_1^{(-)} = 0 \qquad \lambda_1 = 0 \tag{33b}$$

and from (28) we have

$$|\psi_{n=0}\rangle = |0\rangle|-1\rangle \tag{34a}$$

$$\mathcal{E}_0 = -\omega. \tag{34b}$$

Formulae (30)-(34) represent the solution to problem (25). Now we consider two important particular cases.

(i) Let $\lambda_1 = 0$. Then for $n \geqslant 2$ from (31) it follows that

$$y_2 = -4\lambda_1^2/\lambda_2 n + o(\lambda_1^2) \qquad y_{1,3} = \pm\lambda_2[(n-1)/n]^{1/2} + o(1) \qquad \lambda_1 \to 0$$

and for the state vector and energy levels of a system we obtain from (29) and (30)

$$|\psi_n^{(\pm)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm|n-2\rangle \\ 0 \\ |n\rangle \end{pmatrix} \qquad |\psi_n\rangle = \begin{pmatrix} 0 \\ |n-1\rangle \\ 0 \end{pmatrix} \tag{35a}$$

$$\mathscr{E}_n^{(\pm)} = \omega n - \omega \pm \lambda_2\sqrt{n(n-1)} \qquad \mathscr{E}_n = \omega(n-1). \tag{35b}$$

Hereafter the representation

$$|n\rangle|-1\rangle = \begin{pmatrix} 0 \\ 0 \\ |n\rangle \end{pmatrix} \qquad |n\rangle|0\rangle = \begin{pmatrix} 0 \\ |n\rangle \\ 0 \end{pmatrix} \qquad |n\rangle|1\rangle = \begin{pmatrix} |n\rangle \\ 0 \\ 0 \end{pmatrix}$$

is used; $|\psi_1^{(\pm)}\rangle$ and $\mathscr{E}_1^{(\pm)}$ at $\lambda_1 = 0$ are given by (33). At $n = 0$ the solution is defined by (34).

(ii) Let $\lambda_2 = 0$, which corresponds to the standard Jaynes-Cummings model for a three-level atom.

For $n \geqslant 2$

$$y_2 = \frac{2(n-1)}{2n-1}\lambda_2 + o(\lambda_2) \qquad y_{1,3} = \pm\frac{\lambda_1}{n}[2(2n-1)]^{1/2} + o(1) \qquad \lambda_2 \to 0$$

which upon substituting into (29) and (30) gives

$$|\psi_n^{(\pm)}\rangle = \left(\frac{n}{2(2n-1)}\right)^{1/2}\begin{pmatrix} [(n-1)/n]^{1/2}|n-2\rangle \\ \pm[(2n-1)/n]^{1/2}|n-1\rangle \\ |n\rangle \end{pmatrix}$$

$$|\psi_n\rangle = \left(\frac{n-1}{2n-1}\right)^{1/2}\begin{pmatrix} -[n/(n-1)]^{1/2}|n-2\rangle \\ 0 \\ |n\rangle \end{pmatrix} \tag{36a}$$

$$\mathscr{E}_n^{(\pm)} = \omega n - \omega \pm \lambda_1[2(2n-1)]^{1/2} \qquad \mathscr{E}_n = \omega n - \omega. \tag{36b}$$

At $n = 1$ and $n = 0$ we make use of formulae (32) and (34), respectively.

From the general formula (23) we conclude that

$$\dim \mathscr{H}_{N=n-1} = \begin{cases} 3 & n \geqslant 2 \\ 2 & n = 1 \\ 1 & n = 0 \end{cases}$$

which is in full correspondence with the results obtained above. On the basis of the spectrum (32*b*), (34*b*) and (36*b*) for system (25) with $\lambda_2 = 0$ we can obtain the same expressions for the characteristic parameters of an atomic system in the resonance case, as Buck and Sukumar (1984b) have done. The completeness of the system of functions (33*a*)-(35*a*) and (32*a*), (34*a*) and (36*a*) can be easily verified.

4.3. Other models with non-linear interaction

The Hamiltonian

$$\mathscr{H}_r^{k;n} = \omega a^+ a + \omega_0 S_3 + \sum_{j=1}^{[2r/n]} \frac{\lambda_j}{j!}(a^k S_+^n)^j + \text{CC} \tag{37}$$

where k and n are natural numbers, $S_3|m\rangle = m|m\rangle$, $-r \leqslant m \leqslant r$ and $[x]$ is the integer part of x, is an immediate generalisation of the Hamiltonian $\mathscr{H}_r$. Indeed, $\mathscr{H}_r = \mathscr{H}_r^{k=n=1}$. The interaction $\mathscr{H}_r^{k;n}$ is associated with a conserved charge $\hat{N} = a^+ a + (k/n)S_3$, $[\mathscr{H}, \hat{N}] = 0$, which allows us to apply the above method of constructing solutions of the Schrödinger equations, corresponding to definite eigenvalues, to the Hamiltonian (37).

In particular, for the simplest Hamiltonian (37)

$$\mathscr{H}_{r=1}^{k=1;n=2} = \omega a^+ a + \omega_0 S_3 + \lambda a S_+^2 + \lambda a^+ S_-^2 \qquad \hat{N} = a^+ a + \tfrac{1}{2} S_3$$

we obtain for the energy levels, eigenvalues of the charge and dimensionalities of the corresponding subspaces

$$\mathscr{E}_n^{(\pm)} = \omega n - \tfrac{1}{2}\omega \pm [(\tfrac{1}{2}\omega - \omega_0)^2 + 4\lambda^2 n]^{1/2} \qquad N = n - \tfrac{1}{2} \qquad \dim \mathscr{H}_N = 2 \qquad n \geqslant 1$$

$$\mathscr{E}_{n=0} = -\omega_0 \qquad N = -\tfrac{1}{2} \qquad \dim \mathscr{H}_{N=-1/2} = 1$$

$$\mathscr{E}'_n = \omega n \qquad N = n \qquad \dim \mathscr{H}_{N=n} = 1 \qquad n \geqslant 0.$$

5. Conclusions

The generalisations of the Jaynes-Cummings model we have considered represent the simplest solvable models that describe the essential physics of radiation-matter interactions. For instance, these models are capable of describing such interesting effects, now being intensively studied theoretically and experimentally, as vacuum-field Rabi oscillations, the revival and collapse of Rabi oscillations due to the coherent pumping and photon antibunching. Hamiltonians of type (1), (12), (21) and (37) are essentially used for studying multiphoton processes in finite-level systems. In particular, the spectra we have obtained could be applied to study the behaviour of the photon statistics of multiphoton absorption and emission in a two-level atomic system (Shen 1967, Agarwal 1970, Walls 1971, Zubairy and Yeh 1980, Voigt *et al* 1980), two-photon and more general multiphoton lasers (McNeil and Walls 1975, Nayak and Mohanty 1979, Reid *et al* 1981, Zubairy 1982) and Raman and hyper-Raman processes (Simaan 1978, Sainz de los Terreros *et al* 1985). As to other possible trends in studies based on the Jaynes-Cummings model, mention should be made of a recent series of papers (Agarwal and Puri 1986, Puri and Agarwal 1986) devoted to the generalisation of the Jaynes-Cummings model to include the effects of cavity damping.

References

Agarwal G S 1970 *Phys. Rev.* A **1** 1445
Agarwal G S and Puri R R 1986 *Phys. Rev.* A **33** 1757
Buck B and Sukumar C V 1981a *Phys. Lett.* **81A** 132
—— 1981b *Phys. Lett.* **81A** 211

Buck B and Sukumar C V 1984a *J. Phys. A: Math. Gen.* **17** 885
—— 1984b *J. Phys. A: Math. Gen.* **17** 877
Eberley J H, Narozhny N B and Sanchez-Mondragon J J 1981 *Phys. Rev.* A **23** 236
Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
Lee B S 1973 *J. Phys. C: Solid State Phys.* **6** 2873
McNeil K J and Walls D F 1975 *J. Phys. A: Math. Gen.* **8** 104
Nayak N and Mohanty B K 1979 *Phys. Rev.* A **19** 1204
Puri R R and Agarwal G S 1986 *Phys. Rev.* A **33** 3610
Reid M, McNeil K J and Walls D F 1981 *Phys. Rev.* A **24** 2029
Rupasov V 1982 *JETP Lett.* **36** 115
Sainz de los Terreros L, Santos M and Gonzalez-Diaz P F 1985 *Phys. Rev.* A **31** 1598
Sentitzky I R 1971 *Phys. Rev.* A **3** 421
Shen Y R 1967 *Phys. Rev.* **155** 921
Simaan H D 1978 *J. Phys. A: Math. Gen.* **11** 1799
Voigt H, Bandilla A and Ritze H-H 1980 *Z. Phys.* B **36** 295
Walls D F 1971 *J. Phys. A: Gen. Phys.* **4** 813
Zubairy M S 1982 *Phys. Lett.* **87A** 162
Zubairy M S and Yeh J J 1980 *Phys. Rev.* A **21** 1624